# Quantum spins on star graphs and the Kondo model

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**Abstract**

We study the XX model for quantum spins on the star graph with three legs (i.e., on a $Y$-junction). By performing a Jordan-Wigner transformation supplemented by the introduction of an auxiliary space we find a Kondo Hamiltonian of fermions, in the spin 1 representation of $su(2)$, locally coupled with a magnetic impurity. We also show that it is possible to find a XY model such that - after the Jordan-Wigner transformation - one obtains a quadratic fermionic Hamiltonian directly diagonalizable.

Keywords: Quantum spin model; Kondo model; Jordan-Wigner transformation; Star graph

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# 1 Introduction

Spatial inhomogeneities and their role in the emergence of coherent behaviors at mesoscopic length scales are the subject of a continuing interest. In general, spatial inhomogeneities may be random, due to the presence of disorder or noise, as well as non-random, as a result of an external control on the geometry of the system: in a broad sense, their formation can be dynamically generated or induced through a suitable engineering of the system. As a consequence the effects of spatial inhomogeneities have been investigated in a variety of systems, ranging from pattern formation in systems with competing interactions [1] to Josephson networks with non-random, yet non-translationally invariant architecture [2, 3].

A paradigmatic system in which the effects of spatial inhomogeneities can be studied is provided by spin models: on the one hand, not only do spin Hamiltonians directly describe many phenomena of magnetic systems [4], including the effects of frustration [5], but they are also routinely used to model physical properties of several condensed matter systems. On the other hand, spatial inhomogeneities can be straightforwardly included in spin models, to explore the consequences of the breaking of the translational invariance and the local properties on the length scales of the inhomogeneities [6].

As an example of the application of the study of spatial inhomogeneities in spin systems, we mention the spin chain Kondo effect. The standard Kondo effect arises from the interactions between magnetic impurities and the electrons in a metal and characterized by a net increase at low temperature of the resistance [7, 8, 9]. The Kondo effect has been initially observed for metals, like copper, in which magnetic atoms, like cobalt, are added: however, interest in the Kondo physics persisted also because it is can be studied with quantum dots [10, 11]. The universal low-energy/long-distance physics of the Kondo model can be simulated and studied by a magnetic impurity coupled to a gapless antiferromagnetic one-dimensional chain [12]: the rationale is that the free electron Kondo problem may be described by a one-dimensional model since only the $s$-wave part of the electronic wavefunction is affected by the Kondo coupling [13]. Using the spin chain version of the Kondo problem, a characterization of the Kondo regime using negativity was recently presented [14, 15].

Another reason of interest for introducing spatial inhomogeneities in spin models defined on networks is given by the study of the effects of the topology of the graph on the properties of the system and of the breaking of integrability. As a main example, consider a quantum (classical) spin model which is integrable in one (two) dimensions. Techniques have been developed to deal with open boundary conditions [16], as for free boundaries described by algebraic curves [17]. However, if some vertices of the graph on which the spins are located have a number of nearest neighbours larger than all others, then integrability is in general broken. One can see this by considering a one-dimensional quantum model which can be solved by a Jordan-Wigner (JW) transformation [18]: intersecting the chain at one site with a finite or infinite number of other chains the usual JW transformation on the spin variables will produce a fermionic model which is neither quadratic nor local. We recall that the two-dimensional classical Ising model at finite temperature can be solved by writing its partition functions in terms of a suitable quantum spin model on the chain which is solved by JW transformation [19, 20]. Therefore, finding an effective way of performing a JW transformation in non trivial graphs amounts to the possibility of studying and possibly solving the Ising model in some non trivial (non two-dimensional) lattices [21].

In this paper we study the XX model on a star graph obtained by merging three chains and we show by an exact mapping that it is equivalent to a generalized Kondo model, where the JW fermions enter locally and quadratically, and are coupled to a magnetic impurity. To obtain this result, we found convenient to supplement the application of the standard JW transformation with the introduction of an auxiliary space. The procedure of adding auxiliary sites to perform a JW transformation has been recently used to study multidimensional systems in [22, 23]. In our case, it is the use of this auxiliary space which allows us to get a Hamiltonian that is both quadratic and local in the JW fermions.

There are several reasons for our choice of the XX model on a star graph. On the one hand, we study the XX model since we are motivated by the need of emphasize the main point of our construction in the simplest

case: for the XX model in a chain, the JW transformation gives rise to free fermions (our construction can be extended to other spin models solvable by JW transformations).

On the other hand, we decide to restrict ourselves to the study of a star graph with three legs for a twofold reason: first, it is the simplest graph which can be constructed by merging a finite number of chains and having a finite number of vertices (three in our construction, see Figure 1) with coordination number $z = 3$ different and larger than the others (having $z = 2$, with the sites at the boundaries of the chain having $z = 1$). Second, the star graph (alias, the $Y$-junction) has been deeply studied in different contexts from different point of views: for three Tomonaga-Luttinger liquids (TLL) crossing at a point new attractive fixed points emerge [24, 25, 26]. Regular networks of TLL, with each node described by a unitary scattering matrix, were also studied [27], obtaining the same renormalization group equations derived for a single node coupled to several semi-infinite 1D wires [24]. The transport through one-dimensional TLL coupled together at a single point has been also studied [28]. $Y$-junctions of superconducting Josephson junctions were as well analyzed: for suitable values of the control parameters an attractive finite coupling fixed point is found [26], displaying an emerging two-level quantum system with enhanced coherence [29]. Star graphs were studied also in connection with bosonic models: properties of an ideal gas of bosons on a star graph were investigated in [30, 31] and the possible experimental realization with ultracold bosons was discussed in [31]. The dynamics of one-dimensional Bose liquids in $Y$-junctions and the reflection at the center of the star was studied, discussing the emergence of a repulsive fixed point [32]. Finally, we mention that the study of different theories on a graph and, particularly, on a star graph is a very active field of research: for example, for the Laplacian operator (also called quantum graphs) [33, 34, 35, 36, 37], for the Dirac operator [38, 39], for classical field theories and soliton theories [40, 41, 42] and for quantum field theories [43].

The plan of the paper is the following: in section 2, we introduce the XX model on a star graph, and we perform the JW transformation needed to obtain a fermionic Hamiltonian. The usefulness to add auxiliary sites is motivated, and the obtained Kondo Hamiltonian derived and discussed. In section 3, we show that it is possible to find an XY model such that after the JW transformation one obtains a quadratic fermionic Hamiltonian directly diagonalizable. Finally, our conclusions are presented in section 4.

# 2 The XX model on a star graph

In this section, we want to obtain fermionic Hamiltonians from quantum spin models on a star graph by using a JW transformation. In particular, we point out the importance to add an auxiliary site to obtain a fermionic Hamiltonian: we show that to solve this model is equivalent to solve a generalized Kondo model. The treatment is explicitly done for the XX model to emphasize our construction in the simplest case, although the procedure can be used to study other models on the star graph.

## 2.1 The model and the Jordan-Wigner transformation

We introduce in this section the XX model on a three-leg star graph. The graph we consider is illustrated in Figure 1 and it is made of three chains of length $L$, each one having vertices labeled by $1, \cdots, L$; the sites 1 of each of the three chains are connected between them. In each vertex of the graph (having $3L$ vertices) are defined the Pauli matrices $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. As usual we use the notation $\sigma^{\pm} = \frac{1}{2}(\sigma^x \pm i\sigma^y)$.

The XX model is described by the following quantum Hamiltonian acting on the Hilbert space $(\mathbb{C}^2)^{\otimes 3L}$:

$$\widetilde{H}_3^{XX} = \sum_{j=1}^{L-1} \sum_{\alpha=1}^{3} \sigma_\alpha^+(j)\sigma_\alpha^-(j+1) + \rho \sum_{\alpha=1}^{3} \sigma_\alpha^+(1)\sigma_{\alpha+1}^-(1) + h.c. \,, \tag{2.1}$$

where $\sigma_\alpha^\pm(j)$ stands for the matrix $\sigma^\pm$ acting on the $\alpha^{th}$ chain (with $\alpha = 1, 2, 3$) and on the $j^{th}$ site from the vertex (the labeling of the sites is plotted in Fig. 1). In equation (2.1) we have used the convention $\sigma_4^\pm(1) := \sigma_1^\pm(1)$. The parameter $\rho$ (in general complex) entering in the definition of the Hamiltonian (2.1) is a free parameter allowing one to modify the coupling constant at the center of the star graph. In particular, for $\rho = 0$, one retrieves three independent XX models on segments with free (open) boundaries.

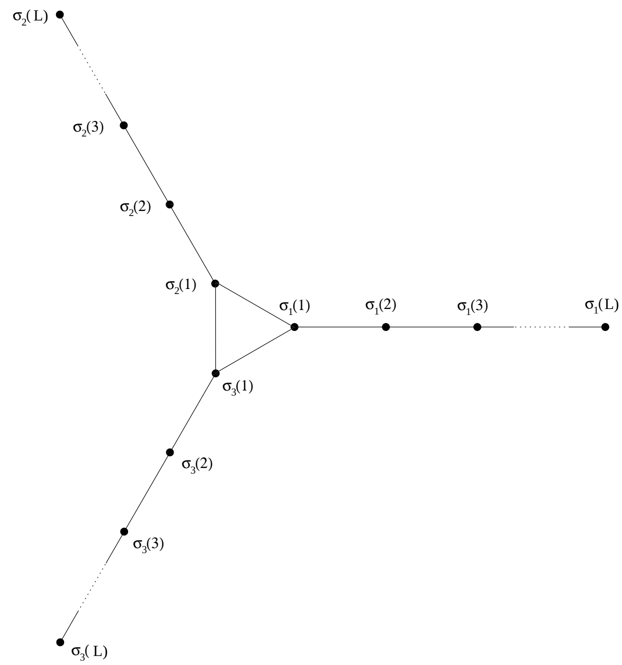


Figure 1: The three-leg star graph and the labeling of the vertices.

At this point, we arrive at the main ingredient of our construction. To perform a JW transformation, we introduce, instead of $\widetilde{H}_3^{XX}$, a slightly different Hamiltonian $H_3^{XX}$ acting on the Hilbert space $\mathbb{C}^2 \otimes (\mathbb{C}^2)^{\otimes 3L}$ and defined by the following rules: $H_3^{XX}$ acts as $\widetilde{H}_3^{XX}$ on the last $3L$ $\mathbb{C}^2$-spaces and trivially on the first $\mathbb{C}^2$-space. The added space (in comparison with the Hilbert space of $\widetilde{H}_3^{XX}$) is denoted 0 and is called *auxiliary space*. We can write the link between both Hamiltonians as follows:

$$H_3^{XX} = Id(0) \otimes \widetilde{H}_3^{XX} \,, \tag{2.2}$$

where $Id(0)$ is the 2 by 2 identity matrix acting on the auxiliary space. Notice that $H_3^{XX}$ has exactly the same spectrum as $\widetilde{H}_3^{XX}$ but with a degeneracy multiplied by 2. Although the addition of this auxiliary site is trivial for the quantum spin model, we will see that it allows one to perform the JW transformation to get a fermionic model (see also the discussion at the end of this section to motivate why this auxiliary space seems necessary). The use of auxiliary sites to perform a JW transformation has been recently used in multidimensional spin systems [22, 23].

Let us introduce the following operators:

$$\eta^x = \sigma^x(0) \prod_{k=1}^{L} \sigma_2^z(k)\sigma_3^z(k) \,, \quad \eta^y = \sigma^y(0) \prod_{k=1}^{L} \sigma_1^z(k)\sigma_3^z(k) \,, \quad \eta^z = \sigma^z(0) \prod_{k=1}^{L} \sigma_1^z(k)\sigma_2^z(k) \,. \tag{2.3}$$

These operators share the same relations of the the Pauli matrices, since they satisfy, for $a, b = x, y, z$,

$$\eta^{a\dagger} = \eta^a \quad , \quad \{\eta^a, \eta^b\} = 2\delta_{ab} \quad \text{and} \quad \eta^x\eta^y = i\eta^z \,, \tag{2.4}$$

where $\{.,.\}$ stands for the anti-commutator.

The JW transformation we use is, for $j = 1, 2, \dots, L$, defined by

$$c_1(j) = \eta^x \left(\prod_{k=1}^{j-1} \sigma_1^z(k)\right) \sigma_1^-(j) \ , \quad c_2(j) = \eta^y \left(\prod_{k=1}^{j-1} \sigma_2^z(k)\right) \sigma_2^-(j) \ , \quad c_3(j) = \eta^z \left(\prod_{k=1}^{j-1} \sigma_3^z(k)\right) \sigma_3^-(j) \ . \tag{2.5}$$

An important point is that $\eta^x$ commutes with $\prod_{k=1}^{j-1} \sigma_1^z(k)\ \sigma_1^-(j)$ but anti-commutes with $\prod_{k=1}^{j-1} \sigma_2^z(k)\ \sigma_2^-(j)$. As usual, we define $c_\alpha(j)^\dagger$ as the conjugate transpose of $c_\alpha(j)$ (for $\alpha = 1, 2, 3$ and $j = 1, 2, \dots, L$).

It is possible to show that $c_\alpha(j)$ and $c_\alpha(j)^\dagger$ are fermionic operators. They indeed satisfy for $\alpha, \beta = 1, 2, 3$ and $j, k = 1, 2, \dots, L$ the following anti-commutation relations:

$$\{c_\alpha(j)\ ,\ c_\beta(k)\} = 0 \ , \qquad \{c_\alpha(j)^\dagger\ ,\ c_\beta(k)^\dagger\} = 0 \ , \qquad \{c_\alpha(j)\ ,\ c_\beta(k)^\dagger\} = \delta_{\alpha,\beta}\delta_{jk} \ . \tag{2.6}$$

The explicit proof of these relations is based on the following statements: firstly, the last two factors in equations (2.5) are the usual JW transformations [18] and provide the anti-commutation between terms in the same leg. The anti-commutation between different legs is provided by the first factor (see equation (2.4)). This factor may be viewed as a Klein factor which has been used extensively in literature: it allows one to define correctly the bosonization [44] (see also [45, 46, 47]) and has been used in different contexts, including the 2-channel Kondo model [48], quantum wire junctions described by coupled TLL [49, 25] or the free quantum field theory on a star graph [50].

We conclude this subsection by observing that for $a = x, y, z$, $\beta = 1, 2, 3$ and $j = 1, 2, \dots, L$, the following relations hold:

$$[\eta^a\ ,\ c_\beta(j)] = 0 \quad \text{and} \quad [\eta^a\ ,\ c_\beta(j)^\dagger] = 0 \ . \tag{2.7}$$

### 2.2 The Kondo model

By using the result of Section 2.1, it is possible to construct a model equivalent to $H_3^{XX}$ expressed in terms of fermions. Indeed, by using relations (2.5), we can express the Hamiltonian $H_3^{XX}$ in terms of the operators $c_\alpha(j)$, $c_\alpha(j)^\dagger$ and $\eta^a$ as follows

$$H_3^{XX} = -\sum_{j=1}^{L-1}\sum_{\alpha=1}^{3} c_\alpha(j)^\dagger c_\alpha(j+1) + i\rho\Big(\eta^z\ c_1(1)^\dagger c_2(1) + \eta^x\ c_2(1)^\dagger c_3(1) + \eta^y\ c_3(1)^\dagger c_1(1)\Big) + h.c. \tag{2.8}$$

To write more compactly the Hamiltonian (2.8) we introduce $\{S^x, S^y, S^z\}$, the $su(2)$ generators in the 3-dimensional representation, as

$$S^x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \ , \qquad S^y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \ , \qquad S^z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{2.9}$$

Then, for $\rho \in \mathbb{R}$, the Hamiltonian (2.8) becomes

$$H_3^{XX} = -\sum_{j=1}^{L-1}\sum_{\alpha=1}^{3}\Big(c_\alpha(j)^\dagger c_\alpha(j+1) + c_\alpha(j+1)^\dagger c_\alpha(j)\Big) - \rho \sum_{a=x,y,z}\ \sum_{\alpha,\beta=1,2,3} \eta^a\ c_\alpha(1)^\dagger (S^a)_{\alpha\beta} c_\beta(1) \ . \tag{2.10}$$

Finally, by using vectorial notation, this Hamiltonian may be rewritten in a more compact way as follows

$$H_3^{XX} = -\sum_{j=1}^{L-1}\Big(\boldsymbol{c(j)^\dagger c(j+1)} + \boldsymbol{c(j+1)^\dagger c(j)}\Big) - \rho\ \boldsymbol{\eta}\ .\ \boldsymbol{c(1)^\dagger S c(1)} \ , \tag{2.11}$$

where

$$\boldsymbol{c(j)}^\dagger = (c_1(j)^\dagger, c_2(j)^\dagger, c_3(j)^\dagger) \ , \quad \boldsymbol{\eta} = (\eta^x, \eta^y, \eta^z) \ , \quad \boldsymbol{S} = \begin{pmatrix} S^x \\ S^y \\ S^z \end{pmatrix} \ . \tag{2.12}$$

The expression (2.10) allows us to interpret the Hamiltonian $H_3^{XX}$ as the Hamiltonian of free fermions coupled with a magnetic impurity. More precisely, it is a $su(2)$ Kondo model with free fermions in the spin 1 representation and a magnetic impurity in the fundamental representation.

Apparently, although the literature about the Kondo model is huge, this particular case, to our knowledge, has not been studied previously. Indeed, the historical Kondo model [7] - studied using, for example, perturbation theory [51], numerical renormalization group [52] or exact methods [53, 54] - corresponds to spin 1/2 free fermions coupled with a spin 1/2 impurity. Different generalizations have been introduced and studied: spin $S$ impurities [55], the $su(N)$ version, so-called the Coqblin-Schrieffer model [56], or the multi-channel Kondo models [57]. The more general models studied deals with the multi-channel $su(N)$ fermions with a spin $S$ impurity [58, 59], considering the fundamental representations for the fermions. Let us emphasize that the model (2.10) obtained in this section, as it is written, is not directly a multi-channel Kondo model.

We conclude this section by commenting on the JW transformation we used: one may think of using the JW transformation (3.2) instead of (2.5), replacing $\eta^a$ by $\sigma^a(0)$. However, in this case, although we get a Hamiltonian similar to (2.8), the fermionic operators obtained do not satisfy the commutation relations (2.7) with the $\eta$'s. Therefore, we can not anymore to interpret the model as a Kondo model. Furthermore one could also think of different JW transformations without adding an auxiliary space. However, such transformations have their drawbacks: for example, if one performs the transformations (2.5) without the first factor (the $\eta$'s), one obtains a quadratic Hamiltonian, but quadratic in hardcore bosons, not fermions. Alternatively, one could consider a JW transformations following a "spiral" according

$$\mathfrak{c}_{3(j-1)+\alpha} = \left(\prod_{k=1}^{j-1} \sigma_1^z(k)\sigma_2^z(k)\sigma_3^z(k)\right)\left(\prod_{\beta=1}^{\alpha-1} \sigma_\beta^z(j)\right)\sigma_\alpha^-(j) \quad \text{for } j = 1, 2, \dots, L \ ; \quad \alpha = 1, 2, 3 \ . \tag{2.13}$$

The operators $\mathfrak{c}_j$ are fermionic, however the Hamiltonian finally obtained is not quadratic in these operators. With these two examples of transformation, one sees that the reason for which the standard JW transformation cannot work for a star graph is that there is no natural order on it. Of course, this problem would generically appear for any graph, except for the circle and the segment where it works.

# 3 Free fermions on a star graph and associated spin chains

In this section, we investigate if it is possible to find a XY model on a star graph which, after a JW transformation gives only a quadratic fermionic Hamiltonian. In comparison with the previous section, we allow ourselves to modify the interaction between the spins near the vertex.

## 3.1 Link between Hamiltonians

Since it would be very cumbersome to explore all the interactions between spins at the vertex to find the ones providing a quadratic fermionic Hamiltonian, we proceed in the following way: we start from a quadratic fermionic Hamiltonian on the star graph and we perform a JW transformation, obtaining a quantum spin model on the star graph. In this section, we restrict ourselves to the three-leg star graph (one may extend easily the obtained results).

We start from the following quadratic fermionic Hamiltonian on a three-leg star graph

$$\widetilde{H}_3^{QF} = \sum_{j=1}^{L-1}\sum_{\alpha=1}^{3}\Big(d_\alpha(j)^\dagger d_\alpha(j+1) - \gamma d_\alpha(j)d_\alpha(j+1)\Big) + i\sum_{\alpha=1}^{3}\Big(a_\alpha d_\alpha(1)^\dagger d_{\alpha+1}(1) + b_\alpha d_\alpha(1)d_{\alpha+1}(1)\Big) + h.c.\,, \tag{3.1}$$

where $\gamma$, $a_\alpha$ and $b_\alpha$ are coupling constants, $d_\alpha(j)$ and $d_\alpha(j)^\dagger$ are fermionic operators and we have used the conventions $d_4(1) := d_1(1)\,,\ d_4^\dagger(1) := d_1^\dagger(1)$. As in the previous section, instead of $\widetilde{H}_3^{QF}$ we consider the Hamiltonian $H_3^{QF} = Id(0)\otimes\widetilde{H}_3^{QF}$ which trivially acts on a supplementary $\mathbb{C}^2$-space, the auxiliary space denoted by 0. Then, we perform the following JW transformation:

$$d_1(j) = \sigma^x(0)\prod_{k=1}^{j-1}\sigma_1^z(k)\ \sigma_1^-(j)\ ,\ d_2(j) = \sigma^y(0)\prod_{k=1}^{j-1}\sigma_2^z(k)\ \sigma_2^-(j)\ ,\ d_3(j) = \sigma^z(0)\prod_{k=1}^{j-1}\sigma_3^z(k)\ \sigma_3^-(j)\ . \tag{3.2}$$

with $j = 1,2,\dots,L$. A straightforward computation shows that the R.H.S. of (3.2) are fermionic operators (notice the differences with the previous Jordan-Wigner transformations (2.5)).

By using the transformations (3.2), the quadratic fermionic Hamiltonian becomes the following quantum spin chain

$$H_3^{QF} = -\sum_{x=1}^{L-1}\sum_{\alpha=1}^{3}\big(\sigma_\alpha^+(x)\sigma_\alpha^-(x+1) + \gamma\sigma_\alpha^-(x)\sigma_\alpha^-(x+1)\big) - H_V + h.c. \tag{3.3}$$

where

$$\begin{aligned} H_V \ =\ & \sigma^z(0)\big(a_1\ \sigma_1^+(1)\sigma_2^-(1) + b_1\ \sigma_1^-(1)\sigma_2^-(1)\big) + \sigma^x(0)\big(a_2\ \sigma_2^+(1)\sigma_3^-(1) + b_2\ \sigma_2^-(1)\sigma_3^-(1)\big) \\ & + \sigma^y(0)\big(a_3\ \sigma_3^+(1)\sigma_1^-(1) + b_3\ \sigma_3^-(1)\sigma_1^-(1)\big) \end{aligned} \tag{3.4}$$

This shows that it is possible to obtain a XY model on a three-leg star graph which is equivalent to a quadratic fermionic Hamiltonians. Notice, however, that the interactions between spins at the center of the star graph are not of the XY type and involve three spins.

### 3.2 Solution for $a_\alpha = a$ and $b_\alpha = \gamma = 0$

The Hamiltonian $\widetilde{H}_3^{QF}$ given in relation (3.1) is a quadratic fermionic Hamiltonian. Therefore, it can be diagonalized by usual procedures [60, 61]. Evidently, this result provides also the spectrum of the Hamiltonian (3.3) since it is the same spectrum with all the degeneracies multiplied by two.

To present a specific example, we consider in the following the case $a_\alpha = a \in \mathbb{R}$, $b_\alpha = 0$ and $\gamma = 0$ (for $\alpha = 1,2,3$). We can rewrite the Hamiltonian (3.1) as

$$\widetilde{H}_3^{QF} = \sum_{i,j=1}^{3L} d_i^\dagger A_{ij} d_j\ , \tag{3.5}$$

where we have changed the numeration $d_\alpha(j)\to d_{(\alpha-1)L+j}$ and the entries of the matrix $A$ are 0 everywhere except

$$A_{j,j+1} = 1 = A_{j+1,j} \quad \text{where} \quad j = 1,\dots,L-1,L+1,\dots,2L-1,2L+1,\dots,3L-1 \tag{3.6}$$

$$A_{1,L+1} = A_{L+1,2L+1} = A_{2L+1,1} = ia \quad \text{and} \quad A_{1,2L+1} = A_{L+1,1} = A_{2L+1,L+1} = -ia\ . \tag{3.7}$$

To diagonalize $\widetilde{H}_3^{QF}$ one has to diagonalize the $3L$ times $3L$ matrix $A$: it is possible to show that the eigenvalues of $A$ are the roots of the following three polynomials

$$U_L(\lambda/2) \pm a\sqrt{3}\ U_{L-1}(\lambda/2) = 0 \ \text{ or } \ U_L(\lambda/2) = 0 \tag{3.8}$$

where $U_L(x)$ is the Chebyshev polynomials of the second kind of degree $L$. Let us remark that, for $a = 0$, we get three times the same equation which is expected since the system becomes three identical decoupled systems.

The Hamiltonian then becomes

$$\widetilde{H}_3^{QF} = \sum_{k=1}^{3L} |\lambda_k| \left( \xi_k^\dagger \xi_k - \frac{1}{2} \right) , \tag{3.9}$$

where $\xi_k$ are fermionic operators and $\{\lambda_k\}$ is the set of the $3L$ solutions of (3.8). Therefore, the spectrum of $\widetilde{H}_3^{QF}$ is the following set of $2^{3L}$ elements

$$\left\{ \frac{1}{2} \sum_{k=1}^{3L} \epsilon_k |\lambda_k| \text{ such that } \epsilon_k = \pm \right\} \tag{3.10}$$

and the eigenvalue of the ground state is $-\frac{1}{2}\sum_{k=1}^{3L} |\lambda_k|$.

Finally, the problem is solved when the $3L$ solutions of (3.8) are given. It is easy to find them numerically or even analytically. Indeed, for the last equation, its roots are $2cos(k\pi/(L+1))$ with $k = 1, 2, \ldots, L$: we present them in Figure 2 for $a = 1$ and $L = 150$ as a dispersion relation. We remark that the three sets

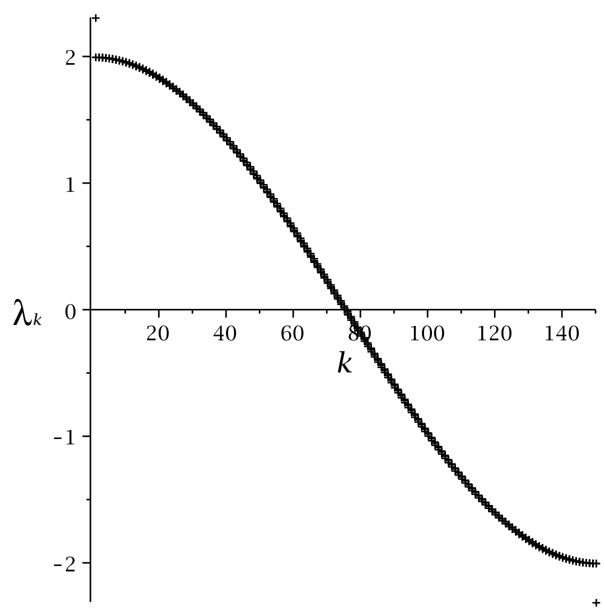


Figure 2: The roots of the polynomials (3.8) for $L = 150$ and $a = 1$.

of solutions are very similar (in Figure 2, these three sets of roots are superimposed). The main difference relies on the presence of one isolated point for each of the first two equations. As shown in subsection 3.1, this result provides the spectrum for the quantum spin model (3.3).

# 4 Conclusions

In this paper we studied the XX model for quantum spin model on a three-leg star graph: we showed that by introducing an auxiliary space and performing a Jordan-Wigner transformation, the model is equivalent

to a generalized Kondo Hamiltonian in which the free fermions, in the spin 1 representation of $su(2)$, are coupled with a magnetic impurity. We also showed that it is possible to find a XY model such that - after the Jordan-Wigner transformation - one obtains a quadratic fermionic Hamiltonian directly diagonalizable.

We observe that we may think of different generalizations of our method. Indeed, the XX Hamiltonian (2.1) can be replaced by the anisotropic XY model with a transverse magnetic field. We can also replace the three-leg star graph by a more complicated graph (e.g., with a different number of legs or with a comb-like graph). Since the spectrum of the Ising model in a transverse magnetic field is directly related to the partition function of the classical Ising model in a corresponding two-dimensional graph [19], the explicit solution of the resulting Kondo Hamiltonian would be relevant to study the classical Ising model in non trivial geometries. We note, however, that even in the case of the XX model on three-leg star graph studied in this paper, the obtained generalized Kondo model appears to be not studied previously and non trivial to be solved. We hope that the results of the present paper will motivate future studies for the Kondo model with fermions in different representations.

We also observe that, since the Hamiltonian (2.1) can be mapped on a Kondo model, it is possible to conclude that its continuous limit is not given by free fermions on a star graph, but by a conformal field theory of Gross-Neveu type, as expected for a Kondo model. It would be of great interest to identify exactly this field theory.

The mapping presented in this paper between the XX quantum spin model on the star graph and the Kondo model illustrates that the introduction of a non trivial topology, even locally, can provide new interesting physical phenomena in comparison to models on the line or on the circle. At the same time, our results show that one may also think to use the XX model on a star graph to realize (or simulate) a Kondo model.

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